On near polygons all whose hexes are dual polar spaces

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Abstract

One of the most fundamental results in the theory of regular near polygons is the result that every regular near $2d$-gon, $d \geq 3$, whose parameters s, t, t_i, $i \in \{0, 1, \ldots, d\}$, satisfy $s, t_2 \geq 2$ and $t_3 = t_2^2 + t_2$ is a dual polar space. The proof of that theorem heavily relies on Tits' theory of buildings, in particular on Tits' strong results on covering of chamber systems. In this paper, we give an alternative proof which only employs geometrical and algebraic combinatorial arguments.

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1 Introduction

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *near $2d$-gon* for some $d \in \mathbb{N}$ if the collinearity graph Γ of $\mathcal{S}$ has diameter d and if for every point-line pair (x, L), there exists a unique point $\pi_L(x)$ on L that is nearest to x with respect to the distance in Γ. A near 0-gon is a point and a near 2-gon is a line. A near quadrangle having a pair of disjoint lines is also called a *generalized quadrangle* [11]. Every near quadrangle is either a generalized quadrangle or a *degenerate generalized quadrangle*. The latter is a point-line geometry of diameter 2 that has some distinguished point x such that every point $y \neq x$ is collinear with x and incident with a unique line (containing x). A near polygon is a near $2d$-gon for some $d \in \mathbb{N}$. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours.

A set X of points of a near polygon $\mathcal{S}$ is called a *subspace* if every line having two of its points in X has all its points in X. Having a nonempty subspace X, we denote by $\widetilde{X}$ the subgeometry of $\mathcal{S}$ defined on the point set X by those lines of $\mathcal{S}$ that have all their points in X. A set X of points is called *convex* if every point on a shortest path between two points of X is also contained in X. If X is a nonempty convex subspace of a near polygon, then $\widetilde{X}$ clearly is a near polygon. If $\widetilde{X}$ is a generalized quadrangle, then X is called a *quad*.

With every polar space Π of rank $n \geq 1$ in the sense of Tits [14, Chapter 7], there is associated a dual polar space Δ of rank n. This is the point-line geometry whose points and lines are the maximal and next-to-maximal singular subspaces of Π, with incidence being reverse containment. By definition, a dual polar space of rank 0 is a point. Every dual polar space of rank n is a near $2n$-gon. If F is a convex subspace of a dual polar space, then also $\widetilde{F}$ is a dual polar space.

If x and y are two points of a dual polar space or a dense near polygon at distance i from each other, then x and y are contained in a unique convex subspace of diameter i. This convex subspace is a quad if $i = 2$ and a *hex* if $i = 3$.

The most fundamental characterization result of dual polar spaces (in terms of near polygons) is the following result due to Cameron [6].

Theorem 1.1. *The dual polar spaces are precisely the near polygons that satisfy the following two properties:*

(1) *Every two points at distance* 2 *are contained in a quad.*

(2) *For every point-quad pair* (x, Q)*, there is a unique point in* Q *nearest to* x*.*

Theorem 1.1 can be proved in an entirely geometrical way, see [6] and Section 8.3 of [8]. Another characterization result of (finite) dual polar spaces is the following result due to Brouwer and Cohen [3].

Theorem 1.2. *Let* $\mathcal{S}$ *be a finite near* $2d$*-gon,* $d \geq 3$*, satisfying the following two properties:*

(1) *Every line of* $\mathcal{S}$ *has at least three points and every two points at distance* 2 *have at least three common neighbours (in particular,* $\mathcal{S}$ *is dense).*

(2) *If* H *is a hex, then* $\widetilde{H}$ *is a dual polar space (of rank* 3*).*

Then $\mathcal{S}$ *itself is also a dual polar space.*

Theorem 1.2 has some applications to regular near polygons. A near polygon is said to have *order* (s, t) if every line is incident with precisely $s + 1$ points and if every point is incident with exactly $t + 1$ lines. A finite near $2d$-gon with $d \geq 2$ is called *regular* if there exist constants s, t, t_i, $i \in \{0, 1, \ldots, d\}$, such that $\mathcal{S}$ has order (s, t) and if x, y are two points at distance i from each other, then $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = t_i + 1$. Obviously, $t_0 = -1$, $t_1 = 0$ and $t_d = t$. The collinearity graphs of regular near polygons provide one of the main families of distance-regular graphs, see Chapter 6 of [4].

From Theorem 1.2, the following can be deduced.

Corollary 1.3. *If* $\mathcal{S}$ *is a finite regular near* $2d$*-gon,* $d \geq 3$*, with parameters* s, t, t_i*,* $i \in \{0, 1, \ldots, d\}$*, such that* $s, t_2 \geq 2$ *and* $t_3 = t_2^2 + t_2$*, then* $\mathcal{S}$ *is a dual polar space.*

Proof. By Theorem 1.2, it suffices to show that $\widetilde{H}$ is a dual polar space for every hex H of $\mathcal{S}$. The hex H is a regular near hexagon with parameters s', t', t'_i, $i \in \{0,1,2,3\}$, such that $s' = s$, $t' = t_3$ and $t'_i = t_i$ for every $i \in \{0,1,2,3\}$. The facts that $s' \geq 2$ and $t'_2 \geq 1$ imply by [13, Proposition 2.5] that any two points of $\widetilde{H}$ at distance 2 from each other are contained in a quad. Under the assumptions that $s' \geq 2$ and $t'_2 \geq 1$, the condition that $t' = (t'_2)^2 + t'_2$ implies by [5, Lemma 25(iii)] that for every point-quad pair (x, Q) of $\widetilde{H}$, there exists a unique point in Q nearest to x. Theorem 1.1 then implies that $\widetilde{H}$ is a dual polar space, as we needed to show. □

The proof of Theorem 1.2 in [3] heavily relies on Tits' theory of buildings, in particular on Tits' strong results on coverings of chamber systems [15]. As (regular) near polygons are usually studied by means of geometrical and (algebraic) combinatorial techniques, it might therefore be desirable to have a more accessible proof for researchers working in that field. The alternative proof of Theorem 1.2 that will be given here only employs geometrical and algebraical combinatorial techniques and does not rely on the theory of buildings nor on Tits' results on covering of chamber systems. The proof which can be found in Section 5 indeed relies on the theory of distance-regular graphs and on some structural properties (Lemma 4.40) of a family of near polygons that we will derive in Section 4 by entirely geometrical tools.

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2 Preliminaries

In this section, we fix some notations and recall some results from the theory of near polygons that will be freely used in Sections 3 and 4. Most of these results are basic and can be found in the standard references [5], [7, Chapter 1], [8, Chapter 8], [13] on the topics.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon. If x and y are two points of $\mathcal{S}$, then $\mathrm{d}(x, y)$ denotes the distance between x and y in the collinearity graph Γ of $\mathcal{S}$. If $\mathrm{d}(x, y) = 1$, then we also write $x \sim y$. If x is a point and X a nonempty set of points, then we define $\mathrm{d}(x, X) := \min\{\mathrm{d}(x, y) \,|\, y \in X\}$. If X_1, X_2 are two nonempty sets of points, then we define $\mathrm{d}(X_1, X_2) := \min\{\mathrm{d}(x_1, x_2) \,|\, x_1 \in X_1 \text{ and } x_2 \in X_2\}$. If O is a point or a nonempty set of points, then $\Gamma_i(O)$ with $i \in \mathbb{N}$ denotes the set of points at distance i from O.

Two lines L_1 and L_2 of $\mathcal{S}$ at distance $k = \mathrm{d}(L_1, L_2)$ from each other are called *parallel* if $\mathrm{d}(x_1, L_2) = \mathrm{d}(x_2, L_1) = k$ for every $x_1 \in L_1$ and every $x_2 \in L_2$. If F is a convex subspace of $\mathcal{S}$ and $x \in \Gamma_1(F)$, then there exists a unique point $\pi_F(x) \in F$ collinear with x and $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every $y \in F$. If F is a convex subspace and x, y are two collinear points of $\Gamma_1(F)$ such that the line xy does not meet F, then $\pi_F(x) \sim \pi_F(y)$. Moreover, the unique line though $\pi_F(x)$ and $\pi_F(y)$ coincides with $\pi_F(xy) := \{\pi_F(z) \,|\, z \in xy\}$ and is parallel with (and at distance 1 from) xy.

If $O_1, O_2, \ldots, O_k$ is a collection of $k \geq 1$ points or nonempty sets of points, then $\langle O_1, O_2, \ldots, O_k \rangle$ denotes the intersection of all convex subspaces containing $O_1, O_2, \ldots, O_k$.

This is well-defined as $\mathcal{P}$ is such a convex subspace. Note that $\langle O_1, O_2, \ldots, O_k\rangle$, as intersection of convex subspaces, is again a convex subspace. In fact, it is the smallest convex subspace containing $O_1, O_2, \ldots, O_k$.

Let Γ be a graph of diameter $d \geq 2$. Γ is called *distance-regular* if there exist constants a_i, b_i, c_i ($i \in \{0, 1, \ldots, d\}$) such that for every two vertices x and y at distance $i \in \{0, 1, \ldots, d\}$, we have

$$|\Gamma_i(x) \cap \Gamma_1(y)| = a_i, \qquad |\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i, \qquad |\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i.$$

Obviously, $a_0 = c_0 = b_d = 0$, $c_1 = 1$ and Γ is regular with valency $k = a_0 + b_0 + c_0 = a_1 + b_1 + c_1 = \ldots = a_d + b_d + c_d$.

The regular near $2d$-gons, $d \geq 2$, are precisely those near $2d$-gons for which the collinearity graph Γ is distance-regular. The connection between the parameters s, t, t_i of $\mathcal{S}$ and the parameters a_i, b_i, c_i of Γ is given as follows (with $i \in \{0, 1, \ldots, d\}$):

$$a_i = (s-1)(t_i + 1), \qquad b_i = s(t - t_i), \qquad c_i = t_i + 1.$$

A *polar space of rank* $n \geq 1$ in the sense of Tits [14, Chapter 7] is a pair $\Pi = (X, \Sigma)$, where X is a nonempty set whose elements are called *points* and Σ is a set of subsets of X, called *singular subspaces*, for which the following four axioms are satisfied:

(A) A singular subspace together with the singular subspaces contained in it defines a projective space of dimension $r \leq n-1$.

(B) The intersection of any two singular subspaces is again a singular subspace.

(C) If S is an $(n-1)$-dimensional singular subspace and $x \in X \setminus S$, then there exists a unique $(n-1)$-dimensional singular subspace S' containing x and intersecting S in an $(n-2)$-dimensional singular subspace. Moreover, $S \cap S'$ consists of those elements of S that are contained with x in a singular subspace of dimension 1.

(D) There exist two disjoint singular subspaces of dimension $n-1$.

We note that the projective spaces occurring in (A) are allowed to be degenerate (and so can have two points on a line). As mentioned in Section 1, with Π there is associated a dual polar space whose points and lines are the $(n-1)$- and $(n-2)$-dimensional singular subspaces, with incidence being reverse containment. In the case Π and Δ are finite and Δ satisfies the property that every line is incident with at least three points and every two points at distance 2 have at least three common points, we have the following possibilities for $n \geq 2$ by Tits' classification of polar spaces [14].

Π	Δ	Ambient space	Defining object	e
$W(2n-1,q)$	$DW(2n-1,q)$	$\mathrm{PG}(2n-1,q)$	symplectic polarity	1
$Q(2n,q)$	$DQ(2n,q)$	$\mathrm{PG}(2n,q)$	parabolic quadric	1
$Q^-(2n+1,q)$	$DQ^-(2n+1,q)$	$\mathrm{PG}(2n+1,q)$	elliptic quadric	2
$H(2n-1,q)$, q square	$DH(2n-1,q)$	$\mathrm{PG}(2n-1,q)$	hermitian polarity	$\frac{1}{2}$
$H(2n,q)$, q square	$DH(2n,q)$	$\mathrm{PG}(2n,q)$	hermitian polarity	$\frac{3}{2}$

In each case, the polar space Π is defined by a polarity ζ or a quadric Q of the ambient projective space. The singular subspaces are either totally isotropic with respect to ζ or contained in Q. The associated dual polar space Δ is regular with parameters s, t, t_i, $i \in \{0, 1, \ldots, d\}$, where $s = q^e$, $t_i = \frac{q^i-1}{q-1} - 1$ ($i \in \{0, 1, \ldots, d\}$) and $t = t_d$.

As the intersection of a quad Q and a convex subspace is a convex subspace, it is either Q, a line, a singleton or the empty set. Any two points of a near polygon at distance 2 from each other are thus contained in at most one quad, or equivalently, any two distinct intersecting lines are contained in at most one quad. This implies that the local space $\mathcal{L}_x$ in each point is a partial linear space. This *local space* is defined as the point-line geometry whose points and lines are the lines and quads through x, with incident being containment. In a dense near polygon, every two distinct intersecting lines are contained in a unique quad, implying that every local space is a linear space. In a dual polar space Δ of rank n, every local space is a projective space of dimension $n-1$.

A convex subspace F of a near polygon is called *classical*, if for every point x there exists a unique point $\pi_F(x) \in F$ for which $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every $y \in F$. Every convex subspace of a dual polar space Δ is classical.

Two convex subspaces F_1 and F_2 of diameter $n-1$ of a near $2n$-gon are called *parallel* if each point x of F_i, $i \in \{1, 2\}$, is collinear with a unique point $\pi_{F_{3-i}}(x)$ of F_{3-i}. If this is the case, then $\widetilde{F_1} \cong \widetilde{F_2}$ by [7, Theorem 1.10]. An isomorphism between $\widetilde{F_1}$ and $\widetilde{F_2}$ is given by $F_1 \to F_2 : x \mapsto \pi_{F_2}(x)$; its inverse is given by $F_2 \to F_1 : x \mapsto \pi_{F_1}(x)$.

In the case of a dual polar space Δ of rank n, convex subspaces of diameter $n-1$ are also called *maxes*. If M is a max of Δ and F is a convex subspace of diameter i of meeting M, then either $F \subseteq M$ or $F \cap M$ is a convex subspace of diameter $i-1$. Two disjoint maxes M_1 and M_2 of Δ are always parallel and every point of Δ is contained in a quad that intersects M_1 and M_2 in lines. For every max M of Δ, there exists a max disjoint from M. If F is a convex subspace of diameter i of Δ, then the maximal distance from a point of Δ to F is equal to $n-i$. In particular, if M is a max, then every point has distance at most 1 from M.

If (x, Q) is a point-quad pair of a near polygon $\mathcal{S}$, then at least one of the following cases occurs by [12, Lemma 1.3] (or [7, Theorem 1.22]).

(1) Q contains a unique point $\pi_Q(x)$ nearest to x. In this case, $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_Q(x)) + \mathrm{d}(\pi_Q(x), y)$ for every $y \in Q$.

(2) The points in Q nearest to x form an *ovoid* of Q, i.e. a set of points of Q meeting each line of the generalized quadrangle $\widetilde{Q}$ in a singleton.

(3) The subgraph of the collinearity graph induced on Q is a bipartite graph and the set of points of Q nearest to x is a proper subset of size at least 2 of one of the two maximal cocliques of this bipartite graph.

The point x is called *classical* (*ovoidal*, respectively *thin ovoidal*) with respect to Q depending on whether case (1), (2) or (3) occurs.

3 Quad-closed sets

In this section, we prove some facts on quad-closed sets of points in dual polar spaces that will be useful in Section 4.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near polygon with the property that every two points x_1 and x_2 at distance 2 are contained in a unique quad, necessarily equal to $\langle x_1, x_2 \rangle$. A set X of points of $\mathcal{S}$ is called *quad-closed* if it is a subspace and if for any two points x_1 and x_2 of X at distance 2 from each other, the quad $\langle x_1, x_2 \rangle$ is contained in X. Clearly, $\mathcal{P}$ is a quad-closed set and the intersection of any two quad-closed sets is again quad-closed. If X is a set of points of $\mathcal{S}$, then $\langle X \rangle_q$ denotes the intersection of all quad-closed sets that contain X. Since $\langle X \rangle_q$ is also a quad-closed set, it is the smallest quad-closed set that contains the set X.

Lemma 3.1. *Let Δ be a dual polar space of rank $n \geq 1$, let F be a max of Δ and let L be a line of Δ which intersects F in a point x. Then $\langle F \cup L \rangle_q$ coincides with the whole set of points of Δ.*

Proof. We prove that through every $y \in F$, there exists a line M_y which is contained in $\langle F \cup L \rangle_q$, but not in F. We will prove this by induction on the distance $d(x, y)$. Obviously, the claim is valid if $d(x, y) = 0$ since, in this case, we can take M_y equal to L. Suppose now that $d(x, y) > 0$ and let $z \in F$ be a point collinear with y at distance $d(x, y) - 1$ from x. By the induction hypothesis, there exists a line M_z through z which is contained in $\langle F \cup L \rangle_q$, but not in F. Now, let M_y be a line of the quad $\langle M_z, yz \rangle$ through y distinct from yz. Since $M_z \cup yz \subseteq \langle F \cup L \rangle_q$, the quad $\langle M_z, yz \rangle$ is completely contained in $\langle F \cup L \rangle_q$. In particular, M_y is contained in $\langle F \cup L \rangle_q$. Since $yz = \langle M_z, yz \rangle \cap F$ and $M_y \neq yz$, M_y is not contained in F.

Now, let u be an arbitrary point of Δ. If $u \in F$, then $u \in \langle F \cup L \rangle_q$. Suppose therefore that $u \notin F$, let v be the unique point of F collinear with u and let L_v denote a line through v contained in $\langle F \cup L \rangle_q$, but not in F. Such a line exists by the previous paragraph. If $u \in L_v$, then $u \in \langle F \cup L \rangle_q$. If $u \notin L_v$, then the quad $\langle uv, L_v \rangle$ intersects F in a line M and we have $\langle uv, L_v \rangle = \langle L_v, M \rangle$. Since $L_v \cup M \subseteq \langle F \cup L \rangle_q$, we have $\langle uv, L_v \rangle \subseteq \langle L_v, M \rangle \subseteq \langle F \cup L \rangle_q$. In particular, $u \in \langle F \cup L \rangle_q$. $\square$

We can generalize Lemma 3.1 as follows.

Lemma 3.2. *Let Δ be a dual polar space of rank $n \geq 1$, let X be a nonempty subspace of Δ such that $\widetilde{X}$ is connected. Then $\langle X \rangle_q = \langle X \rangle$.*

Proof. Let k be the diameter of $\langle X \rangle$. We construct by induction a chain $F_0 \subset F_1 \subset \cdots \subset F_k$ of convex subspaces satisfying: (i) F_i, $i \in \{0, 1, \ldots, k\}$, has diameter i; (ii) F_i, $i \in \{0, 1, \ldots, k\}$, is contained in $\langle X \rangle$. These conditions imply that $F_k = \langle X \rangle$.

Let $F_0 = \{y_0\}$ be any singleton contained in X. Suppose that we have defined F_i for a certain $i \in \{0, \ldots, k-1\}$. Then F_i has diameter $i < k$ and is properly contained in $\langle X \rangle$. So, by the connectedness of $\widetilde{X}$, there is a point $y_i \in X \setminus F_i$ which is collinear with some point z_i of F_i. Then we define $F_{i+1} = \langle F_i, z_i y_i \rangle = \langle F_i, y_i \rangle$. Clearly, F_{i+1} has diameter $i+1$ and is contained in $\langle X \rangle$.

Using the just-constructed chain $F_0 \subset F_1 \subset \cdots \subset F_k$, we show that $\langle X \rangle_q = \langle X \rangle$. Obviously, we have $\langle X \rangle_q \subseteq \langle X \rangle = F_k$. In order to prove that $\langle X \rangle_q = \langle X \rangle$, it suffices to prove by induction on $i \in \{0, 1, \ldots, k\}$ that $F_i \subseteq \langle X \rangle_q$. Obviously, this is true if $i = 0$ (since $F_0 = \{y_0\} \subseteq X$) and if $i = 1$ (since $F_1 = y_0 y_1 \subseteq X$). So, suppose that $i \in \{1, 2, \ldots, k-1\}$ and that F_i is contained in $\langle X \rangle_q$. Let L_i be the line $y_i z_i$. Since $F_i \cup \{y_i\} \subseteq \langle X \rangle_q$, we have $F_i \cup L_i \subseteq \langle X \rangle_q$. By Lemma 3.1, $\langle F_i, L_i \rangle_q = \langle F_i, L_i \rangle = F_{i+1}$. Since $F_i \cup L_i \subseteq \langle X \rangle_q$, we necessarily have that $F_{i+1} = \langle F_i, L_i \rangle_q \subseteq \langle X \rangle_q$. $\square$

4 A family of near polygons

In this section, we study in an entirely geometrical way a family of near polygons that include the dense near polygons and the dual polar spaces. One of the results we obtain here (Lemma 4.40) will play an essential role in the proof of Theorem 1.2 that will be given in Section 5.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near $2n$-gon, $n \geq 4$, that satisfies the following three properties:

(P1) For any two points x and y of $\mathcal{S}$, the convex subspace $\langle x, y \rangle$ has diameter $\mathrm{d}(x, y)$.

(P2) Every two points at distance 2 have at least two common neighbours.

(P3) If M is a max of $\mathcal{S}$, then $\widetilde{M}$ is a dual polar space of rank $n-1$.

For near polygons satisfying properties (P1) and (P2), a *max* is defined as a convex subspace of diameter $n-1$, and a *hex* is defined as a convex subspace of diameter 3. Every dense near polygon and every dual polar space satisfies properties (P1) and (P2), see e.g. Sections 6.9 and 8.1 of [8].

Our ultimate goal here will be to show that $\mathcal{S}$ has special subgeometries that are dual polar spaces of rank n, and to determine several properties of these subgeometries. These goals will be achieved in the final lemmas and theorems of this section. First, we need to do some preparatory work.

The first fourteen results that we will derive here (Lemma 4.1 till Lemma 4.14) are valid for general near polygons satisfying properties (P1) and (P2) (without requiring (P3)).

Lemma 4.1. *Every two points at distance* 2 *are contained in a (necessarily unique) quad.*

Proof. Let x and y be two arbitrary points at distance 2 from each other, and put $Q := \langle x, y \rangle$. By property (P1), Q has diameter 2 and so the near quadrangle $\widetilde{Q}$ is either a generalized quadrangle or a degenerate generalized quadrangle. The last possibility cannot occur as otherwise x and y have a unique neighbour, which is in contradiction with property (P2). So, $\widetilde{Q}$ is a generalized quadrangle and Q is a quad. $\square$

Lemma 4.2. (1) *Every two distinct intersecting lines are contained in a unique quad. As a consequence, every local space is linear.*

(2) *Any two parallel lines at distance* 1 *from each other are contained in a unique quad.*

Proof. (1) If L_1 and L_2 are two distinct intersecting lines and x_i with $i \in \{1, 2\}$ is a point of L_i not contained in L_{3-i}, then the quads containing L_1 and L_2 are precisely the quads containing x_1 and x_2, and so there is a unique such quad by Lemma 4.1.

(2) Let L_1 and L_2 be two parallel lines at distance 1 from each other and let L_3 and L_4 be two distinct lines meeting L_1 and L_2 in points. Any quad containing L_1 and L_2 also contains L_3 and so must coincide with the quad $\langle L_1, L_3 \rangle$.

It remains to show that the quad $Q = \langle L_1, L_3 \rangle$ also contains the line L_2. As the point $L_2 \cap L_4$ is collinear with the points $L_2 \cap L_3$ and $L_1 \cap L_4$ of Q, it is contained in Q, implying that also the lines L_2 and L_4 are contained in Q. $\square$

Lemma 4.3. *Every point-quad pair is classical or ovoidal.*

Proof. Suppose Q is a quad and x is a point that is thin-ovoidal with respect to Q. Then $\widetilde{Q}$ is a dual grid. Put $\mathrm{d}(x, Q) = k$. Then $|\Gamma_k(x) \cap Q| \geq 2$, $\Gamma_{k+2}(x) \cap Q \neq \emptyset$ and $C_1 := (\Gamma_k(x) \cap Q) \cup (\Gamma_{k+2}(x) \cap Q)$ and $C_2 := \Gamma_{k+1}(x) \cap Q$ are the two maximal cocliques of Q.

Let $y \in \Gamma_{k+1}(x) \cap Q$, and consider the convex subspace $\langle x, y \rangle$ of diameter $k+1$. Every point of $\Gamma_k(x) \cap Q$ is collinear with y and so is contained in the convex subspace $\langle x, y \rangle$. As $|\Gamma_k(x) \cap Q| \geq 2$ and $\Gamma_k(x) \cap Q$ is a coclique, this implies that the whole quad Q is contained in the convex subspace $\langle x, y \rangle$. So, the convex subspace $\langle x, y \rangle$ of diameter $k+1$ contains points at distance $k+2$ from x (namely the points of $\Gamma_{k+2}(x) \cap Q$), which is impossible. $\square$

Lemma 4.4. *If F is a convex subspace of diameter δ of $\mathcal{S}$, then every shortest path of length $i < \delta$ in F can be extended to a shortest path of length δ in F.*

Proof. Consider a shortest path of length i between two points x and y of F. Let F' denote the convex subspace $\langle x, y \rangle$ of diameter i. As F is a convex subspace, we have $F' = \langle x, y \rangle \subseteq F$. As $i < \delta$, there exists a line $L \subseteq F$ intersecting F' in a singleton $\{z\}$. We suppose that we have chosen the line L in such a way that $\mathrm{d}(y, z)$ is minimal. We show that $z = y$.

Suppose $z \neq y$, and let z' be a point of $\Gamma_1(z)$ at distance $\mathrm{d}(z, y) - 1$ from y. As F' is a convex subspace, we have $zz' \subseteq F'$. As L is not contained in F', the quad $Q = \langle L, zz' \rangle$ intersects F' in zz'. So, any line U of Q through z' distinct from zz' would be contained in F (as $L, zz' \subseteq F$ and F is convex) and intersect F' in z', in contradiction with the minimality of $\mathrm{d}(y, z)$.

So, we have $z = y$. If $u \in L \setminus \{y\}$, then $\mathrm{d}(u, F') = 1$ and $y = \pi_{F'}(u)$. So, $\mathrm{d}(u, x) = \mathrm{d}(u, y) + \mathrm{d}(y, x) = i + 1$. We can thus extend the path to a path of length $i + 1$ in F. A straightforward inductive argument then shows that the path can be extended to a shortest path of length δ in F. $\square$

The following is an immediate consequence of Lemma 4.4.

Corollary 4.5. *If F is a convex subspace of diameter δ, then for every point $x \in F$, there exists a point $y \in F$ at distance δ from x.*

Lemma 4.6. *Every two points at distance δ are contained in a unique convex subspace of diameter δ.*

Proof. Let x and y be two points at distance δ from each other. By property (P1), $\langle x, y \rangle$ is a convex subspace of diameter δ containing x and y. Suppose $F \neq \langle x, y \rangle$ is another convex subspace of diameter δ containing x and y. As $\langle x, y \rangle$ is the smallest convex subspace containing x and y, $\langle x, y \rangle$ is properly contained in F. As F is connected, there exists some point $u \in F \setminus \langle x, y \rangle$ that is collinear with a point $v \in \langle x, y \rangle$. By Corollary 4.5, we know that there exists some point $w \in \langle x, y \rangle$ at distance δ from v. Now, $\mathrm{d}(u, \langle x, y \rangle) = 1$ and v is the unique point of $\langle x, y \rangle$ collinear with u, implying that $\mathrm{d}(u, w) = \mathrm{d}(u, v) + \mathrm{d}(v, w) = \delta + 1$, in contradiction with the fact that the diameter of F is only δ. $\square$

The following can easily be derived from Lemma 4.6 and property (P1).

Corollary 4.7. *Every nonempty convex subspace is of the form $\langle x, y \rangle$ for some points x and y.*

Proof. Suppose F is a convex subspace of diameter δ and x, y are points of F at distance δ from each other. Since there exists a unique convex subspace of diameter δ through x and y, we necessarily have $F = \langle x, y \rangle$. $\square$

Lemma 4.8. *Let F be a convex subspace of diameter δ and L a line intersecting F in a singleton $\{x\}$. Then $\langle F, L \rangle$ is a convex subspace of diameter $\delta + 1$.*

Proof. By Corollary 4.5, there exists a point $y \in F$ at distance δ from x. Let $z \in L \setminus \{x\}$. As $\mathrm{d}(z, F) = 1$ and $x = \pi_F(z)$, we have $\mathrm{d}(z, y) = \mathrm{d}(z, x) + \mathrm{d}(x, y) = \delta + 1$. Since x is on a shortest path from y to z, any convex subspace containing y and z contains x and hence also $\langle y, x \rangle = F$. So, $\langle F, L \rangle$ necessarily coincides with $\langle y, z \rangle$ and is a convex subspace of diameter $\delta + 1$. $\square$

Lemma 4.9. *Let F be a convex subspace of diameter δ and Q a quad intersecting F in a line, then $\langle F, Q \rangle$ is a convex subspace of diameter $\delta + 1$.*

Proof. Let x be an arbitrary point of the line $L := Q \cap F$, and let L' be a line of Q through x distinct from L. As $Q = \langle L, L' \rangle$, we know by Lemma 4.8 that $\langle F, Q \rangle = \langle F, L, L' \rangle = \langle F, L' \rangle$ is a convex subspace of diameter $\delta + 1$. $\square$

Definition. For every two points x and y of $\mathcal{S}$, let $S(x, y)$ denote the set of lines through x containing a point at distance $\mathrm{d}(x, y) - 1$ from y. Obviously, $S(x, y) = \emptyset$ if $x = y$, $S(x, y) = \{xy\}$ if $\mathrm{d}(x, y) = 1$ and $S(x, y)$ is the whole set of lines through x if $\mathrm{d}(x, y) = n$.

Lemma 4.10. *For every two points x and y, $S(x, y)$ is the set of lines through x contained in $\langle x, y \rangle$.*

Proof. Put $\delta = \mathrm{d}(x, y)$. Then $F := \langle x, y \rangle$ is a convex subspace of diameter δ. Since $\widetilde{F}$ is a near polygon, any line L of F through x contains a unique point at minimal distance $\delta - 1$ from y, implying that $L \in S(x, y)$. Conversely, any line K of $S(x, y)$ contains a point at distance $\mathrm{d}(x, y) - 1$ from y and the convexity of $F = \langle x, y \rangle$ then implies that K must be contained in F. □

Lemma 4.11. (1) *For every convex subspace F and every $x \in F$, the set of lines of F through x is a subspace of $\mathcal{L}_x$.*

(2) *For every two points x and y of $\mathcal{S}$, the set $S(x, y)$ is a subspace of $\mathcal{L}_x$.*

Proof. (1) Let L_1 and L_2 be two distinct lines through x contained in F. As F is a convex subspace, the quad $Q = \langle L_1, L_2 \rangle$ must also be contained in F, in particular, every line of Q through x is contained in F. This shows Claim (1).

(2) Claim (2) is a consequence of Claim (1) and Lemma 4.10. □

Definition. The subspace of $\mathcal{L}_x$ determined by the lines of F through x (as in Lemma 4.11(1)) is called the *subspace of $\mathcal{L}_x$ induced by F*.

Lemma 4.12. *Let $x_0, x_1, \ldots, x_\delta$ be a shortest path between two points x_0 and x_δ. Then for every $i \in \{1, 2, \ldots, \delta\}$, the subspace $S(x_0, x_{i-1})$ of $\mathcal{L}_{x_0}$ is a proper subset of $S(x_0, x_i)$.*

Proof. Let L be an arbitrary line of $S(x_0, x_{i-1})$. Then L contains a point at distance $i-2$ from x_{i-1} and thus a point at distance at most (and hence precisely) $i - 1$ from x_i. So, L also belongs to $S(x_0, x_i)$ implying that $S(x_0, x_{i-1}) \subseteq S(x_0, x_i)$. It thus suffices to show that there is some line $L \in S(x_0, x_i)$ not belonging to $S(x_0, x_{i-1})$. In view of Lemma 4.10, we need to show that there is some line L of $\langle x_0, x_i \rangle$ through x_0 not contained in $\langle x_0, x_{i-1} \rangle$. But this follows from Lemma 4.4 by extending a shortest path between x_{i-1} and x_0 to a shortest path of length i in F between x_{i-1} and y. Then $L = x_0 y$ is the required line. □

Lemma 4.13. *Let x be a point of $\mathcal{S}$. Then a subspace S of $\mathcal{L}_x$ is induced by at most one convex subspace through x. If F is such a convex subspace, then $F = \{x\}$ if $S = \emptyset$ and $F = \langle \bigcup_{L \in S} L \rangle$ otherwise.*

Proof. If $S = \emptyset$, then $\{x\}$ is the unique convex subspace through x inducing S. So, we may suppose that $S \neq \emptyset$.

Suppose that S is induced by a convex subspace F through x and let F' denote the convex subspace $\langle \bigcup_{L \in S} L \rangle$. As $\bigcup_{L \in S} L$ is contained in F, we have $F' \subseteq F$. By construction F' also contains all lines of S. Since S is induced by F, we thus see that S is also induced by F'. In order to prove the uniqueness of the convex subspace through x inducing S, it suffices to show that $F = F'$.

Suppose F' is properly contained in F. Let δ' be the diameter of F' and δ the diameter of F. As there exists some line in F intersecting F' in a singleton, we know by Lemma 4.8 that $\delta > \delta'$. By Corollary 4.5, there exists a point y in F' at maximal distance δ' from x. By Lemma 4.4, there exists a point $z \in \Gamma_1(y) \cap F$ at distance $\delta' + 1$ from x. By Lemma 4.10 and the fact that $F' = \langle x, y \rangle$ induces the subspace S of $\mathcal{L}_x$, we have $S = S(x, y)$. By Lemma 4.12 and the fact that $y = \pi_{F'}(z)$ is on a shortest path from $x \in F'$ to z, we know that $S(x, y)$ is properly contained in $S(x, z)$. Since $x, z \in F$ and F is a convex subspace inducing the subspace S of $\mathcal{L}_x$, we see that $S(x, z) \subseteq S$. In summary, we have $S = S(x, y) \subsetneq S(x, z) \subseteq S$, which is clearly an impossibility. So, we indeed must have that $F' = F$. $\square$

The following lemma is even valid in an arbitrary near polygon.

Lemma 4.14. *Let γ be a path of length $k + 1$ connecting two points x and y at distance $k \in \{0, 1, \ldots, n\}$ from each other. Then any point of γ is contained in $\langle x, y \rangle$.*

Proof. Suppose $\gamma : x = x_0, x_1, \ldots, x_{k+1} = y$ and let $l \in \{0, 1, \ldots, k + 1\}$ be the smallest value for which $d(x, x_l) < l$. Then $l \geq 2$ and $d(x, x_{l-1}) = l - 1$. If $d(x, x_l) = l - 2$, then $d(x_l, x_{k+1}) \geq \mathrm{d}(x_0, x_{k+1}) - \mathrm{d}(x_0, x_l) \geq k - l + 2$, contradicting the fact that there exists a path of length $k + 1 - l$ between x_l and x_{k+1}. Hence, $d(x, x_{l-1}) = d(x, x_l) = l - 1$. Let z be the unique point on $x_{l-1}x_l$ at distance $l - 2$ from x. Then $x_{l-1} \neq z \neq x_l$. There is a path γ_1 of length $l-2$ connecting x and z and a path $\gamma_2 : z, x_l, \ldots, x_{k+1}$ of length $k-l+2$ connecting z and x_{k+1}. So, the concatenation of γ_1 and γ_2 is a shortest path (of length $l - 2 + k - l + 2 = k$) between x and y. It follows that $z \in \langle x, y \rangle$ and $x_i \in \langle x, y \rangle$ for every $i \in \{l, l+1, \ldots, k+1\}$. Since $x_{l-1} \in zx_l$, also $x_{l-1} \in \langle x, y \rangle$. Since x_i, $i \in \{0, 1, \ldots, l-1\}$ is contained in a shortest path from x to x_{l-1}, also $x_i \in \langle x, y \rangle$ for every $i \in \{0, 1, \ldots, l - 1\}$. This finishes the proof that every point of γ is contained in $\langle x, y \rangle$. $\square$

From now on, we will rely on property (P3).

Lemma 4.15. *If F is a convex subspace of diameter $\delta \leq n - 1$, then $\widetilde{F}$ is a dual polar space of rank δ.*

Proof. By successively applying Lemma 4.8 , we see that there exists some max M through F. Then F is a convex subspace of $\widetilde{M}$. Since $\widetilde{M}$ is a dual polar space, also $\widetilde{F}$ is a dual polar space, necessarily of rank δ. $\square$

Lemma 4.16. *Every local space of $\mathcal{S}$ is a (possibly degenerate) projective space of dimension at least $n - 1$. All local spaces have the same dimension.*

Proof. Let x be an arbitrary point of $\mathcal{S}$. If Q_1 and Q_2 are two quads through x which intersect in a line, then by Lemmas 4.9 and 4.15 $F := \langle Q_1, Q_2 \rangle$ is a hex for which $\widetilde{F}$ is a dual polar space of rank 3. So, the lines and quads through x contained in F define a projective plane. Hence, every two lines of $\mathcal{L}_x$ that meet in a point are contained in a subgeometry of $\mathcal{L}_x$ that is a projective plane. This implies that $\mathcal{L}_x$ is a projective space.

By Corollary 4.5, there exists a path $x = x_0, x_1, \ldots, x_n$ of length n connecting the point x with a point x_n at maximal distance n from x. For every $i \in \{0, 1, \ldots, n\}$, let S_i denote the subspace of $\mathcal{L}_x$ induced by $\langle x_0, x_i \rangle$. By Lemma 4.12, S_{i-1} is properly contained in S_i for every $i \in \{1, 2, \ldots, n\}$. This implies that the dimension of the projective space $\mathcal{L}_x$ is at least $n-1$.

In order to show that all local spaces have the same dimension, it suffices by the connectedness of $\mathcal{S}$ to prove this for the local spaces $\mathcal{L}_x$ and $\mathcal{L}_y$, where x and y are distinct collinear points. But this follows from the fact that the quotient projective spaces $\mathcal{L}_x/xy$ and $\mathcal{L}_y/xy$ are isomorphic. Indeed, both $\mathcal{L}_x/xy$ and $\mathcal{L}_y/xy$ are isomorphic to the point-line geometry whose points, respectively lines, are the quads, respectively hexes, through xy (natural incidence). $\square$

In the following lemma, we strengthen one of the claims of Lemma 4.13.

Lemma 4.17. *Let x be a point of $\mathcal{S}$ and let S be a subspace of $\mathcal{L}_x$ whose dimension δ is at most $n-2$. Then S is induced by a necessarily unique convex subspace of diameter $\delta+1$ through x.*

Proof. We prove this by induction on δ. Obviously, the lemma holds if $\delta \in \{-1, 0, 1\}$. So, suppose $\delta \geq 2$ and let $S' \subset S$ be a subspace of dimension $\delta - 1$ of $\mathcal{L}_x$. By the induction hypothesis, S' is induced by a necessarily unique convex subspace F' of diameter δ through x. Let L be an arbitrary line of $S \setminus S'$ and let F be the convex subspace $\langle F', L \rangle$ of diameter $\delta+1$ (see Lemma 4.8). Since $\delta+1 \leq n-1$, $\widetilde{F}$ is a dual polar space of rank $\delta+1$ by Lemma 4.15. So, the local space of $\widetilde{F}$ at the point x is a projective space of dimension δ, i.e., the subspace S'' of $\mathcal{L}_x$ induced by F is a subspace of dimension δ of the projective space $\mathcal{L}_x$. Since $S' \cup \{L\} \subseteq S'' \cap S$, S' is a hyperplane of both S, S'' and $\dim(S) = \dim(S'') = \delta$ as subspaces of $\mathcal{L}_x$, we necessarily have $S = S''$. So, the subspace of $\mathcal{L}_x$ induced by F coincides with S. The uniqueness follows from Lemma 4.13. $\square$

Lemma 4.18. *Let x and y be two distinct collinear points of $\mathcal{S}$. Let S_x be a subspace of $\mathcal{L}_x$ containing $L = xy$ and let $\mathcal{Q}$ be the set of all quads Q through L that induce a line of $\mathcal{L}_x$ that is contained in S_x. Let S_y denote the set of lines through y that are contained in some quad of $\mathcal{Q}$. Then S_y is a subspace of $\mathcal{L}_y$ having the same dimension as the subspace S_x of $\mathcal{L}_x$.*

Proof. Let L_1 and L_2 be two distinct lines of S_y. We need to prove that every line L_3 of $\langle L_1, L_2 \rangle$ through y is contained in S_y. If $L \subseteq \langle L_1, L_2 \rangle$, then $\langle L_1, L_2 \rangle \in \mathcal{Q}$ and hence the claim holds. We will therefore suppose that $L \not\subseteq \langle L_1, L_2 \rangle$. Then $F := \langle L, L_1, L_2 \rangle$ is a hex by Lemma 4.8 and $Q_i := \langle L, L_i \rangle \in \mathcal{Q}$, $i \in \{1, 2\}$. By Lemma 4.15, $\widetilde{F}$ is a dual polar space of rank 3 (containing the quads Q_1, Q_2) and so the subspace of $\mathcal{L}_x$ induced by F is a projective plane. Since S_x is a subspace of $\mathcal{L}_x$ and $Q_1, Q_2 \in \mathcal{Q}$, every line of F through x must therefore belong to S_x. This implies that $\langle L, L_3 \rangle \in \mathcal{Q}$ and hence that $L_3 \in S_y$. As said before, this implies that S_y is a subspace of $\mathcal{L}_y$.

In order to prove that S_x and S_y have the same dimension (regarded as subspaces of respectively $\mathcal{L}_x$ and $\mathcal{L}_y$), it suffices to show that the quotient spaces S_x/xy and S_y/xy

have the same dimension. But this is obvious, since each of these quotients is isomorphic to the point-line geometry whose points are the elements of $\mathcal{Q}$ and whose lines are the hexes of $\mathcal{S}$ through xy which induce a subspace of $\mathcal{L}_x$ contained in S_x, or equivalently, a subspace of $\mathcal{L}_y$ contained in S_y. □

Definitions. (1) Let x and y be two distinct collinear points of $\mathcal{S}$, let S_x be a subspace of $\mathcal{L}_x$ containing xy and let S_y be a subspace of $\mathcal{L}_y$ containing xy. If S_y is obtained from S_x in the way as described in Lemma 4.18, then we say that the pair (S_x, S_y) is *compatible*. Clearly, if (S_x, S_y) is compatible, then also (S_y, S_x) is compatible.

(2) Let F be a max and $y \in \Gamma_1(F)$. Then y is collinear with a unique point x of F. Let S'_x be the $(n-2)$-dimensional subspace of $\mathcal{L}_x$ induced by F, let S_x be the $(n-1)$-dimensional subspace of $\mathcal{L}_x$ generated by S'_x and xy, and let S_y be the unique subspace of $\mathcal{L}_y$ such that (S_x, S_y) is compatible. Then we say that S_y is the *subspace of* $\mathcal{L}_y$ *determined by* F. By Lemma 4.18, the dimension of the subspace S_y of $\mathcal{L}_y$ is equal to $n-1$. As S'_x is a hyperplane of S_x (in $\mathcal{L}_x$), we see that the subspace S_y of $\mathcal{L}_y$ consists of all lines through y that are contained in a quad through xy that meets F in a line.

Lemma 4.19. *Let x be a point and S_x a subspace of dimension $n-1$ of $\mathcal{L}_x$. If F_1 and F_2 are two distinct maxes through x such that the set S_i, $i \in \{1,2\}$, of lines of F_i through x is contained in S_x, then $F_1 \cap F_2$ is a convex subspace of diameter $n-2$.*

Proof. By Lemma 4.16, $\mathcal{L}_x$ is a projective space. Since S_x is a subspace of dimension $n-1$ and S_i, $i \in \{1,2\}$, is a subspace of dimension $n-2$ of S_x, we see that the dimension of $S_1 \cap S_2$ is equal to $n-3$. Note that $S_1 \neq S_2$ by Lemma 4.13. As $S_1 \cap S_2$ is induced by the convex subspace $F_1 \cap F_2$, we see that $F_1 \cap F_2$ is a convex subspace of diameter $n-2$ by Lemmas 4.13 and 4.17. □

Lemma 4.20. *If F_1 and F_2 are two disjoint maxes of $\mathcal{S}$ such that $F_2 \subseteq \Gamma_1(F_1)$, then $F_1 \subseteq \Gamma_1(F_2)$. As a consequence, F_1 and F_2 are parallel maxes at distance 1 from each other.*

Proof. (a) We prove that $d(\pi_{F_1}(x), \pi_{F_1}(y)) = d(x,y)$ for any two distinct points x and y of F_2.

If $d(\pi_{F_1}(x), \pi_{F_1}(y)) < d(x,y)$, then there exists a path γ of length $2+d(\pi_{F_1}(x), \pi_{F_1}(y)) \leq d(x,y)+1$ connecting x and y and containing the points $\pi_{F_1}(x)$ and $\pi_{F_1}(y)$. Lemma 4.14 and the fact that F_2 is a convex subspace imply that all points of γ are contained in $\langle x,y\rangle \subseteq F_2$, contradicting the fact that F_1 and F_2 are disjoint.

If $d(x,y) < d(\pi_{F_1}(x), \pi_{F_1}(y))$, then there exists a path γ of length $2+d(x,y) \leq d(\pi_{F_1}(x), \pi_{F_1}(y))+1$ connecting $\pi_{F_1}(x)$ and $\pi_{F_1}(y)$, and containing the points x and y. Lemma 4.14 and the fact that F_1 is a convex subspace imply that all points of γ are contained in $\langle \pi_{F_1}(x), \pi_{F_1}(y)\rangle \subseteq F_1$, contradicting the fact that F_1 and F_2 are disjoint.

Hence, $d(\pi_{F_1}(x), \pi_{F_1}(y)) = d(x,y)$.

(b) By (a), $F'_2 = \pi_{F_1}(F_2)$ is a subspace of $\mathcal{S}$ and the maximal distance between two points of F'_2 is equal to $n-1$. Hence, $\langle F'_2\rangle = F_1$.

(c) We prove that F_2' is quad-closed. Let $x' = \pi_{F_1}(x)$ and $y' = \pi_{F_1}(y)$ be two points of F_2' at distance 2 from each other, where $x, y \in F_2$. We have $d(x, y) = 2$, $d(x, x') = 1$, $d(y, y') = 1$, $d(x, y') = d(x, x') + d(x', y') = 1 + 2 = 3$ and $d(y, x') = d(y, y') + d(y', x') = 1 + 2 = 3$. The hex $\langle x, y' \rangle$ contains the points x' and y since these points are contained on shortest paths between x and y'. Now, $\widetilde{\langle x, y' \rangle}$ is a dual polar space of rank 3 (recall Lemma 4.15) and $\langle x, y \rangle \subseteq F_1$ and $\langle x', y' \rangle \subseteq F_2$ are two disjoint quads of $\widetilde{\langle x, y' \rangle}$. This implies that every point of $\langle x', y' \rangle$ lies at distance 1 from some point of $\langle x, y \rangle$ and that every point of $\langle x, y \rangle$ lies at distance 1 from some point of $\langle x', y' \rangle$. Hence, $\langle x', y' \rangle \subseteq F_2'$.

(d) By Lemma 4.15, $\widetilde{F_1}$ is a dual polar space of rank $n-1$. Since F_2' is a subspace of F_1 such that $\widetilde{F_2'}$ is connected, $\langle F_2' \rangle_q = \langle F_2' \rangle$ by Lemma 3.2. Hence, $F_2' = \langle F_2' \rangle_q = \langle F_2' \rangle = F_1$. This implies that $F_1 \subseteq \Gamma_1(F_2)$. □

Lemma 4.21. *Let F_1 and F_2 be two parallel disjoint maxes at distance 1 from each other. Let $x \in F_2$, let S_x' be the subspace of $\mathcal{L}_x$ induced by F_2 and let S_x be the subspace of $\mathcal{L}_x$ determined by F_1. Then $S_x' \subseteq S_x$.*

Proof. Put $x' := \pi_{F_1}(x)$. Let L be a line of S_x'. Let y be an arbitrary point of $L \setminus \{x\}$ and put $y' := \pi_{F_1}(y)$. Since y' is contained on a shortest path from x' to y, the quad $\langle xx', L \rangle$ intersects F_1 in the line $x'y'$, implying that $L \in S_x$. □

Lemma 4.22. *Let F_1 be a max of $\mathcal{S}$, let $x_2 \in \Gamma_1(F_1)$, put $x_1 := \pi_{F_1}(x_2)$ and let F_2 be a max through x_2 not containing x_1x_2 such that the subspace of $\mathcal{L}_{x_2}$ induced by F_2 is contained in the subspace of $\mathcal{L}_{x_2}$ determined by F_1. Then F_1 and F_2 are parallel maxes at distance 1 from each other.*

Proof. Let S_i, $i \in \{1, 2\}$, denote the subspace of $\mathcal{L}_{x_i}$ induced by F_i. After showing that $F_1 \cap F_2 = \emptyset$ in (a), we prove in (b) that every point of F_i, $i \in \{1, 2\}$, at distance at most $n-2$ from x_i lies at distance 1 from F_{3-i}. By relying on (b), we subsequently show in (c) that the same also holds for points of F_i at maximal distance $n-1$ from x_i.

(a) We prove that $F_1 \cap F_2 = \emptyset$. Suppose $u \in F_1 \cap F_2$. Then any shortest path between $x_2 \in F_2$ and $u \in F_2$ is contained in F_2. Since $u \in F_1$, there exists a shortest path between x_2 and u that contains the point $x_1 = \pi_{F_1}(x_2)$. So, we would have that $x_1 \in F_2$, clearly a contradiction. Hence, $F_1 \cap F_2 = \emptyset$.

(b) Suppose z is a point of F_1 at distance $k \leq n-2$ from x_1. Put $G_1 := \langle x_1, z \rangle$ and $G := \langle G_1, x_1x_2 \rangle$. Then G has diameter $k+1 \leq n-1$ by Lemma 4.8. Let S_i', $i \in \{1, 2\}$, denote the k-dimensional subspace of $\mathcal{L}_{x_i}$ induced by G. Since $G \cap F_1 = G_1$, $\widetilde{S_1 \cap S_1'}$ is the $(k-1)$-dimensional subspace of $\mathcal{L}_{x_1}$ induced by G_1. By Lemma 4.15, $\widetilde{G} = \langle G_1, x_1x_2 \rangle$ is a dual polar space of rank $k+1$ and so the quads of $\widetilde{G}$ through x_1x_2 cover all lines of S_2' and intersect G_1 in lines. This implies that S_2' is contained in the $(n-1)$-dimensional subspace S^* of $\mathcal{L}_{x_2}$ determined by F_1. By Lemma 4.21 and $x_1x_2 \in S_2' \setminus S_2$, S_2 is a hyperplane of S^* not containing S_2'. This implies that $S_2 \cap S_2'$ is a $(k-1)$-dimensional subspace of $\mathcal{L}_{x_2}$. By Lemma 4.17, $S_2 \cap S_2'$ is induced by a unique convex subspace G_2 of diameter k through x_2. As $S_2 \cap S_2' \subseteq S_2$ and $S_2 \cap S_2' \subseteq S_2'$, we know by Lemma 4.13 that G_2 is contained in

F_2 and in G. As $\widetilde{G}$ is a dual polar space of rank $k+1$ and G_1, G_2 are two disjoint maxes of $\widetilde{G}$, every point of G_1 has distance 1 from $G_2 \subseteq F_2$. Hence, $z \in \Gamma_1(F_2)$.

So, every point of F_1 at distance at most $n-2$ from x_1 is collinear with some point of F_2. Every line of F_1 through x_1 is thus parallel and at distance 1 from a line of F_2 through x_2. By considering suitable quads through x_1x_2, we thus see that the subspace of $\mathcal{L}_{x_1}$ induced by F_1 is contained in the subspace of $\mathcal{L}_{x_1}$ determined by F_2. Similarly as above, one then proves that every point of F_2 at distance at most $n-2$ from x_2 is collinear with some point of F_1.

(c) Suppose z is a point of F_1 at distance $n-1$ from x_1 and let $z_1 \in F_1$ denote a point collinear with x_1 at distance $n-2$ from z. Let z_2 denote the unique point of F_2 collinear with z_1. Then $z_2 \sim x_2$. We prove that the subspace of $\mathcal{L}_{z_2}$ induced by F_2 is contained in the subspace of $\mathcal{L}_{z_2}$ determined by F_1.

So, let L be an arbitrary line through z_2 contained in F_2. By (b), every point of L lies in $\Gamma_1(F_1)$. So, the quad $\langle L, z_1z_2 \rangle$ intersects F_1 in the line $\pi_{F_1}(L)$. This implies that L is indeed contained in the subspace of $\mathcal{L}_{z_2}$ determined by F_1.

Now, by applying (b) to the tuple (F_1, z_2, z_1, F_2) (instead of (F_1, x_2, x_1, F_2)), we see that every point of F_1 at distance at most $n-2$ from z_1 is contained in $\Gamma_1(F_2)$. In particular, $z \in \Gamma_1(F_2)$.

In a similar way, one proves that every point of F_2 at distance $n-1$ from x_2 is contained in $\Gamma_1(F_1)$.

(d) By (b) and (c), $F_1 \subseteq \Gamma_1(F_2)$ and $F_2 \subseteq \Gamma_1(F_1)$. So, F_1 and F_2 are parallel maxes at distance 1 from each other. $\square$

Definition. If F_1 and F_2 are two parallel maxes at distance 1 from each other, then $\Omega(F_1, F_2)$ denotes the set of points at distance at most 1 from F_1 and F_2.

Lemma 4.23. *If F_1 and F_2 are parallel maxes at distance 1 from each other, then $\Omega(F_1, F_2)$ is a subspace.*

Proof. Clearly, $\Omega(F_1, F_2) = S_1 \cap S_2$, where S_i, $i \in \{1, 2\}$, denotes the set of points at distance at most 1 from F_i. It suffices to prove that each S_i, $i \in \{1, 2\}$, is a subspace. So, fix $i \in \{1, 2\}$ and let x and y be two distinct collinear points of S_i. If the line xy meets F_i, then the line xy is contained in S_i. If the line xy is disjoint from F_i, we then know that $xy \subseteq \Gamma_1(F_i)$ is parallel and at distance 1 from the line $\pi_{F_i}(x)\pi_{F_i}(y)$. So, xy is contained in $\Gamma_1(F_i) \subseteq S_i$. $\square$

Lemma 4.24. *Let F_1 and F_2 be two parallel maxes at distance 1 from each other and let L be a line meeting F_1 and F_2. Then every quad through L which meets F_1 in a line also meets F_2 in a line.*

Proof. Put $L \cap F_i = \{x_i\}$, $i \in \{1, 2\}$. Let Q be a quad through L which intersects F_1 in a line K_1. Let $y_1 \in K_1 \setminus \{x_1\}$ and let y_2 denote the unique point of F_2 collinear with y_1. Then $x_2 \sim y_2$ and y_2 is contained in a shortest path from x_2 to y_1. Since $x_2, y_1 \in Q$, also $y_2 \in Q$ and hence the line x_2y_2 is contained in Q. So, Q intersects F_2 in a line. $\square$

Lemma 4.25. *Let F_1 and F_2 be two parallel maxes at distance 1 from each other. Then the points of $\Omega(F_1, F_2)$ are precisely the points of $\mathcal{S}$ which are contained in some quad that intersects F_1 and F_2 in lines.*

Proof. If Q is a quad intersecting F_1 and F_2 in lines, then every point of Q has distance at most 1 from F_1 and F_2 and hence is contained in $\Omega(F_1, F_2)$.

Conversely, suppose that x is a point which has distance at most 1 from F_1 and F_2. If $x_1 := \pi_{F_1}(x)$ and $x_2 := \pi_{F_2}(x)$ are collinear, then x is contained in the line x_1x_2 and hence also in any quad through x_1x_2 which intersects F_1 (and hence also F_2) in a line. Suppose therefore that x_1 and x_2 are not collinear. Then $\mathrm{d}(x_1, x_2) = 2$. Let x_i', $i \in \{1, 2\}$, denote the unique point of F_i at distance 1 from x_{3-i}. As $2 = \mathrm{d}(x_{3-i}, x_i) = \mathrm{d}(x_{3-i}, x_i') + \mathrm{d}(x_i', x_i) = 1 + \mathrm{d}(x_i', x_i)$, the points x_i' and x_i are collinear. The quad through the parallel lines x_1x_1' and x_2x_2' contains x and intersects F_1 and F_2 in lines. $\square$

Lemma 4.26. *Let F_1 and F_2 be two parallel maxes at distance 1 from each other and let $x \in F_1$. Then the set of lines through x contained in $\Omega(F_1, F_2)$ is precisely the $(n-1)$-dimensional subspace S of $\mathcal{L}_x$ determined by F_2.*

Proof. Put $y := \pi_{F_2}(x)$. Notice that $xy \subseteq \Omega(F_1, F_2)$ and $xy \in S$.

Suppose $L \neq xy$ is a line through x contained in $\Omega(F_1, F_2)$. Then $L \subseteq \Gamma_1(F_2)$ and so L projects to the line $\pi_{F_2}(L)$ of F_2. Since the quad $\langle L, xy \rangle$ meets F_2 in the line $\pi_{F_2}(L)$, we have $L \in S$.

Conversely, suppose that $L \neq xy$ belongs to S. Then the quad $\langle L, xy \rangle$ meets F_2 in a line implying that $L \subseteq \Gamma_1(F_2)$. Hence, $L \subseteq \Omega(F_1, F_2)$. $\square$

Lemma 4.27. *Let F_1 and F_2 be two parallel maxes at distance 1 from each other. Let $x_1 \in F_1$ and let S_{x_1} be the $(n-1)$-dimensional subspace of $\mathcal{L}_{x_1}$ determined by F_2. If F is a convex subspace of diameter at most $n-1$ through x_1 which induces a subspace of $\mathcal{L}_{x_1}$ that is completely contained in S_{x_1}, then $F \subseteq \Omega(F_1, F_2)$.*

Proof. By Lemmas 4.13 and 4.17, the convex subspace F is contained in a max which induces a subspace of $\mathcal{L}_{x_1}$ that is completely contained in S_{x_1}. So, it suffices to prove the lemma in the case that F is a max. By Lemma 4.15, we then know that $\widetilde{F}$ is a dual polar space.

Let x_2 be the unique point of F_2 collinear with x_1. We distinguish three cases:

(1) $F = F_1$. Then obviously $F \subseteq \Omega(F_1, F_2)$.

(2) $F \neq F_1$ and F does not contain the line x_1x_2. Then $F \subseteq \Gamma_1(F_2)$ by Lemma 4.22. By Lemma 4.19, $F \cap F_1$ is a max of the dual polar space $\widetilde{F}$ and so every point of F has distance at most 1 from $F \cap F_1$. It follows that $F \subseteq \Omega(F_1, F_2)$.

(3) F contains the line x_1x_2. Then $F \cap F_i$, $i \in \{1, 2\}$, induces a subspace of dimension $n-3$ of $\mathcal{L}_{x_i}$ (as intersection of two subspaces of dimension $n-2$ in a subspace of diameter $n-1$). By Lemmas 4.13 and 4.17, $F \cap F_1$ and $F \cap F_2$ are disjoint maxes of the dual polar space $\widetilde{F}$. It follows that every point of F has distance at most 1 from $F_1 \cap F$ and $F_2 \cap F$. $\square$

Lemma 4.28. *Let F_1 and F_2 be two parallel maxes at distance 1 from each other. If $x \in F_1$ and if $y \in \Omega(F_1, F_2)$ lies at distance at most $n-1$ from x, then $\langle x, y\rangle \subseteq \Omega(F_1, F_2)$.*

Proof. Suppose first that $y \in F_1$. Then $\langle x, y\rangle \subseteq F_1 \subseteq \Omega(F_1, F_2)$.

Suppose now that $y \notin F_1$. Let z denote the unique point of F_1 collinear with y and let S_z denote the $(n-1)$-dimensional subspace of $\mathcal{L}_z$ determined by F_2. Since $y \in \Omega(F_1, F_2)$, $zy \in S_z$ by Lemma 4.26. By Lemma 4.21, the subspace of $\mathcal{L}_z$ induced by $\langle x, z\rangle \subseteq F_1$ is contained in S_z. Since $\widetilde{\langle x, y\rangle}$ is a dual polar space (Lemma 4.15) and $\langle x, y\rangle = \langle\langle x, z\rangle, zy\rangle$, the subspace of $\mathcal{L}_z$ induced by $\langle x, y\rangle$ is generated by zy and the subspace of $\mathcal{L}_z$ induced by $\langle x, z\rangle$ and is therefore contained in S_z. By Lemma 4.27, $\langle x, y\rangle \subseteq \Omega(F_1, F_2)$. □

Lemma 4.29. *Let F_1 and F_2 be two parallel maxes of $\mathcal{S}$ at distance 1 from each other, let G_1 be a convex subspace of diameter $i \leq n-2$ of $\widetilde{F_1}$ and put $G_2 := \pi_{F_2}(G_1)$. Then $\langle G_1, G_2\rangle$ is a convex subspace of diameter $i+1$.*

Proof. Let $x_1 \in G_1$ and put $x_2 := \pi_{F_2}(x_1) \in G_2$. Then $F = \langle G_1, x_1x_2\rangle$ is a convex subspace of diameter $i+1$ by Lemma 4.8. The map $L \mapsto \pi_{F_2}(L)$ defines a bijection between the set $\mathcal{A}_1$ of lines of G_1 through x_1 and the set $\mathcal{A}_2$ of lines of G_2 through x_2. Every quad through x_1x_2 containing a line of $\mathcal{A}_1$ is contained in F and contains the corresponding line of $\mathcal{A}_2$ by Lemma 4.24. So, F contains all lines of $\mathcal{A}_2$ and thus also the convex subspace G_2 by Lemma 4.13. So, $F = \langle G_1, x_1x_2\rangle \subseteq \langle G_1, G_2\rangle \subseteq F$, implying that $\langle G_1, G_2\rangle = F$ has diameter $i+1$. □

Lemma 4.30. *Let F_1 and F_2 be two parallel maxes of $\mathcal{S}$ at distance 1 from each other. Let G_1 and H_1 be two disjoint maxes of $\widetilde{F_1}$, and put $G_2 := \pi_{F_2}(G_1)$, $H_2 := \pi_{F_2}(G_2)$. Then $G := \langle G_1, G_2\rangle$ and $H := \langle H_1, H_2\rangle$ are two parallel maxes at distance 1 from each other.*

Proof. By Lemma 4.29, we know that G and H are maxes of $\mathcal{S}$. By Lemma 4.15, we know that $\widetilde{F_1}$, $\widetilde{F_2}$, $\widetilde{G}$ and $\widetilde{H}$ are dual polar spaces. Let x be an arbitrary point of H_1 and put $y := \pi_{G_1}(x)$, $z := \pi_{F_2}(x) = \pi_{H_2}(x)$. Let S'_x and S''_x denotes the subspaces of $\mathcal{L}_x$ induced by respectively H_1 and H. Then S''_x is the subspace of $\mathcal{L}_x$ generated by S'_x and xz and does not contain xy. Let S_x denote the subspace of $\mathcal{L}_x$ determined by G. As $\widetilde{F_1}$ is a dual polar space, the quads of F_1 through xy intersect G_1 and H_1 in lines, implying that $S'_x \subseteq S_x$. As $z = \pi_{F_2}(x) \in H_2$ is collinear with the point $\pi_{F_2}(y)$ of $G_2 \subseteq G$, we see that the line xz is contained in a quad through xy that meets G in the line $\pi_{F_2}(y)y$. So, also $xz \in S_x$. The subspace S''_x of $\mathcal{L}_x$ generated by S'_x and xz is thus contained in S_x. As $xy \notin S''_x$, Lemma 4.22 implies that G and H are two parallel maxes at distance 1 from each other. □

Lemma 4.31. *Let F_1 and F_2 be two parallel maxes of $\mathcal{S}$ at distance 1 from each other. Let F be a max intersecting F_1 and F_2 in convex subspaces of diameter $n-2$. Then every point of $\Omega(F_1, F_2)$ has distance at most 1 from F.*

Proof. Let G_i, $i \in \{1, 2\}$, be the convex subspace $F_i \cap F$ of diameter $n-2$. Let $x \in \Omega(F_1, F_2)$ and let Q be a quad through x intersecting F_1 in a line L_1 and F_2 in a line L_2 (recall Lemma 4.25). Then $\pi_{F_2}(G_1) = G_2$ and $\pi_{F_2}(L_1) = L_2$. We consider three cases.

(1) $L_1 \subseteq G_1$. Then L_1 and hence also F contains a point at distance at most 1 from x.

(2) $L_1 \cap G_1$ is a singleton $\{y\}$. Then $L_2 \cap G_2$ is the singleton $\{z\}$, where $z = \pi_{F_2}(y)$. The line yz is contained in the quad Q and in F, implying that yz and hence also F contains a point at distance at most 1 from x.

(3) $L_1 \cap G_1 = \emptyset$. Then also $L_2 \cap G_2 = \emptyset$. Put $K_1 = \pi_{G_1}(L_1)$ and $K_2 = \pi_{F_2}(L_2)$. Let Q_i, $i \in \{1, 2\}$, be the quad $\langle K_i, L_i \rangle$. By Lemma 4.29, $H := \langle Q_1, Q_2 \rangle = \langle K_1, L_1, K_2, L_2 \rangle$ is a hex. Note that $Q = \langle L_1, L_2 \rangle$ and $\langle K_1, K_2 \rangle \subseteq F$ are two disjoint quads in H. As $\widetilde{H}$ is a dual polar space of rank 3 (Lemma 4.15), the quad $\langle K_1, K_2 \rangle$ and hence also F contains a point at distance 1 from $x \in Q$. $\square$

The following is an immediate consequence of Lemma 4.31.

Corollary 4.32. *Let F_1 and F_2 be two parallel maxes of $\mathcal{S}$ at distance 1 from each other. Let F_1' and F_2' be two other parallel maxes of $\mathcal{S}$ at distance 1 from each other. If $F_i \cap F_j'$ is a convex subspace of diameter $n-2$ for all $i, j \in \{1, 2\}$, then $\Omega(F_1, F_2) = \Omega(F_1', F_2')$.*

Lemma 4.33. *Let F_1 and F_2 be two parallel maxes of $\mathcal{S}$ at distance 1 from each other and let x be an arbitrary point of $\Omega(F_1, F_2)$. Then there exist two parallel maxes F_1' and F_2' of $\mathcal{S}$ at distance 1 from each other such that $x \in F_1'$ and $\Omega(F_1', F_2') = \Omega(F_1, F_2)$.*

Proof. Recall that $\widetilde{F_1}$ and $\widetilde{F_2}$ are dual polar spaces by Lemma 4.15. By Lemma 4.25, there exists a quad Q containing x intersecting F_1 in a line L_1 and F_2 in a line L_2. Let G be a max of $\widetilde{F_1}$ containing L_1 and G' a max of $\widetilde{F_1}$ disjoint from G. Put $F_1' := \langle G, \pi_{F_2}(G) \rangle$ and $F_2' := \langle G', \pi_{F_2}(G') \rangle$. As $L_1 \subseteq G$ and $L_2 = \pi_{F_2}(L_1) \subseteq \pi_{F_2}(G)$, we have $Q = \langle L_1, L_2 \rangle \subseteq F_1'$. By Lemma 4.30, F_1' and F_2' are two parallel maxes of $\mathcal{S}$ at distance 1. Moreover, $x \in Q \subseteq F_1'$ and by Corollary 4.32 we know that $\Omega(F_1', F_2') = \Omega(F_1, F_2)$. $\square$

The following is an immediate corollary of Lemmas 4.26 and 4.33.

Corollary 4.34. *Let F_1 and F_2 be two parallel maxes of $\mathcal{S}$ at distance 1 from each other and let x be an arbitrary point of $\Omega(F_1, F_2)$. Then the lines through x contained in $\Omega(F_1, F_2)$ determine an $(n-1)$-dimensional subspace of $\mathcal{L}_x$.*

The following is an immediate consequence of Lemmas 4.28 and 4.33.

Corollary 4.35. *Let F_1 and F_2 be two parallel maxes of $\mathcal{S}$ at distance 1 from each other. If $x, y \in \Omega(F_1, F_2)$ lie at distance at most $n-1$ from each other, then $\langle x, y \rangle \subseteq \Omega(F_1, F_2)$.*

Lemma 4.36. *Let F_1 and F_2 be two parallel maxes of $\mathcal{S}$ at distance 1 from each other and let x and y be two points of $\Omega(F_1, F_2)$. Then the distance $d'(x, y)$ between x and y in the geometry $\widetilde{\Omega(F_1, F_2)}$ equals the distance $d(x, y)$ between x and y in $\mathcal{S}$.*

Proof. Note that $\mathrm{d}'(x,y) \geq \mathrm{d}(x,y)$. By Corollary 4.35, the claim of the lemma is true if $\mathrm{d}(x,y) \leq n-1$. If $\mathrm{d}(x,y) = n$, then by considering a line L through y contained in $\Omega(F_1,F_2)$ and taking the unique point $z \in L$ at distance $n-1$ from x, we see that $\mathrm{d}'(x,z) = n-1$ and hence that $\mathrm{d}'(x,y) = n$. □

Lemma 4.37. *If F_1 and F_2 are two parallel maxes at maximal distance 1 from each other, then $\widetilde{\Omega(F_1,F_2)}$ is a near $2n$-gon satisfying (P1), (P2) which is quad-closed.*

Proof. Lemmas 4.23 and 4.36 imply that $\widetilde{\Omega(F_1,F_2)}$ is a near polygon. This near polygon is a near $2n$-gon as the maximal distance between two points of $\Omega(F_1,F_2)$ is equal to n. The remaining claims follows from Corollary 4.35, which states that $\langle x,y\rangle \subseteq \Omega(F_1,F_2)$ for any two points x and y of $\Omega(F_1,F_2)$ at distance at most $n-1$ from each other. □

Lemma 4.38. *Let F_1 and F_2 be two parallel maxes of $\mathcal{S}$ at distance 1 from each other. Then $\widetilde{\Omega(F_1,F_2)}$ is a dual polar space of rank n.*

Proof. By Lemma 4.37, we know that $\widetilde{\Omega(F_1,F_2)}$ is a near $2n$-gon satisfying (P1) and (P2). If $\widetilde{\Omega(F_1,F_2)}$ is not a dual polar space, then there exists by Theorem 1.1 and Lemmas 4.1, 4.3 an ovoidal point-quad pair (x,Q) (in both $\widetilde{\Omega(F_1,F_2)}$ and $\mathcal{S}$). Put $d(x,Q)=k$ and let $z \in Q \setminus \Gamma_k(x)$. Any line of Q through z belongs to $S(z,x)$ and is thus contained in $\langle x,z\rangle$. We thus have $\langle x,Q\rangle = \langle x,z\rangle$ and so $\langle x,Q\rangle$ is a convex subspace of diameter $k+1$ of $\mathcal{S}$. Since x is ovoidal with respect to Q, $\widetilde{\langle x,Q\rangle}$ is not a dual polar space. By Lemma 4.15, $k+1=n$ or $k=n-1$. Now, let $y \in \Gamma_{n-1}(x) \cap Q$. As no line of $S(y,x)$ can be contained in Q, we have $\langle x,y\rangle \cap Q = \{y\}$. By Corollary 4.34, the set of lines of $\mathcal{S}$ through y contained in $\Omega(F_1,F_2)$ is a subspace S of dimension $n-1$ of $\mathcal{L}_y$. By Corollary 4.35, $\langle x,y\rangle \subseteq \Omega(F_1,F_2)$. So, the $(n-2)$-dimensional subspace S_1 of $\mathcal{L}_y$ induced by $\langle x,y\rangle$ is contained in S. The line S_2 of $\mathcal{L}_y$ induced by Q is also contained in S. Since $\langle x,y\rangle \cap Q = \{y\}$, the subspaces S_1 and S_2 of $\mathcal{L}_y$ are disjoint. So, S which contains $S_1 \cup S_2$ must have dimension at least n, a contradiction. □

Definition. Let X be a subspace of $\mathcal{S}$. Then $\widetilde{X}$ is called a *special subgeometry* of $\mathcal{S}$ if the following properties are satisfied: (1) $\widetilde{X}$ is a dual polar space of rank n; (2) the distance $d(x_1,x_2)$ between two points x_1 and x_2 in the geometry $\widetilde{X}$ coincides with the distance between x_1 and x_2 in the geometry $\mathcal{S}$; (3) if $x_1,x_2 \in X$ lie at distance at most $n-1$ from each other, then $\langle x_1,x_2\rangle \subseteq X$. In particular, X is a quad-closed set of points of $\mathcal{S}$.

The following is the main theorem of this section.

Theorem 4.39. (1) *If F_1 and F_2 are parallel maxes at distance 1 from each other, then $\widetilde{\Omega(F_1,F_2)}$ is a special subgeometry of $\mathcal{S}$.*

(2) *If $\widetilde{X}$ is a special subgeometry for some subspace X of $\mathcal{S}$, then there exist two parallel maxes F_1 and F_2 at distance 1 from each other such that $X = \Omega(F_1,F_2)$.*

(3) *Let x and y be two points of $\mathcal{S}$ at distance n from each other and let L be a line through y. Then there exists a unique special subgeometry of $\mathcal{S}$ which contains the point x and the line L.*

Proof. (1) This is an immediate consequence of Corollary 4.35 and Lemmas 4.36, 4.38.

(2) Let F_1 and F_2 be two disjoint maxes of $\widetilde{X}$. Then F_i, $i \in \{1, 2\}$, is also a max of $\mathcal{S}$. For, if u_i and v_i are two points of F_i at distance $n-1$ from each other, then $\langle u_i, v_i \rangle \subseteq X$ implies that $F_i = \langle u_i, v_i \rangle$. If $x \in \widetilde{X}$, then x is contained in a quad of $\mathcal{S}$ meeting F_1 and F_2 in lines and hence is contained in $\Omega(F_1, F_2)$. Conversely, if $x \in \Omega(F_1, F_2)$, then Lemma 4.25 implies that x is contained in a quad Q which meets F_1 and F_2 in lines. The quad Q is also a quad of $\widetilde{X}$ and hence $x \in X$. This proves that $X = \Omega(F_1, F_2)$.

(3) We prove that there exists at least one special subgeometry of $\mathcal{S}$ which contains the point x and the line L. Let z denote the unique point of L at distance $n-1$ from x. Let S_y denote the subspace of $\mathcal{L}_y$ determined by the max $F_1 := \langle z, x \rangle$ and let F_2 denote a max through y not containing yz which induces a subspace of $\mathcal{L}_y$ contained in S_y. By Lemma 4.22, F_1 and F_2 are parallel maxes at distance 1 from each other. By (1), $\widetilde{\Omega(F_1, F_2)}$ is a special subgeometry which satisfies the required properties.

Conversely, suppose that X is a subspace of $\mathcal{S}$ such that $\{x\} \cup L \subseteq X$ and $\widetilde{X}$ is a special subgeometry of $\mathcal{S}$. Since $\widetilde{X}$ is a special subgeometry of $\mathcal{S}$, $\langle x, z \rangle \subseteq X$. By Lemma 3.1 and the fact that X is a quad-closed set of $\mathcal{S}$, we have $X = \langle \langle x, z \rangle \cup L \rangle_q$. So, X is uniquely determined and necessarily coincides with $\Omega(F_1, F_2)$. $\square$

We prove three additional structural properties of $\mathcal{S}$.

Lemma 4.40. *If $\mathcal{S}$ is not a dual polar space, then for every point x of $\mathcal{S}$, $\mathcal{L}_x$ is a projective space of dimension at least $2n-1$.*

Proof. We first show that all local spaces of $\mathcal{S}$ have dimension at least n. We subsequently use this to show that all local spaces of $\mathcal{S}$ have dimension at least $2n-1$.

Since $\mathcal{S}$ is not a dual polar space, there exists by Theorem 1.1 and Lemmas 4.1, 4.3 an ovoidal point-quad pair (y, Q). If $k := d(y, Q)$ and $z \in Q \setminus \Gamma_k(y)$, then similarly as in the proof of Lemma 4.38, we have $\langle y, Q \rangle = \langle y, z \rangle$ is a convex subspace of diameter $k+1$. Since y is ovoidal with respect to Q, $\widetilde{\langle y, Q \rangle}$ is not a dual polar space. By Lemma 4.15, $k+1 = n$, i.e. $k = n-1$. Now, let $z \in \Gamma_{n-1}(y) \cap Q$. Then $\langle y, z \rangle \cap Q = \{z\}$. The set S_1 of lines of $\mathcal{S}$ through z contained in $\langle y, z \rangle$ is a subspace of dimension $n-2$ of $\mathcal{L}_z$. The set S_2 of lines of $\mathcal{S}$ through z contained in Q is a subspace of dimension 1 of $\mathcal{L}_z$. Since $\langle y, z \rangle \cap Q = \{z\}$, we have $S_1 \cap S_2 = \emptyset$. Hence, $\mathcal{L}_z$ has dimension at least n. By Lemma 4.16, every local space of $\mathcal{S}$ has dimension at least n.

Let u be a point of $\mathcal{S}$ opposite to x, let L_1 be a line through x and let X_1 denote the unique subspace of $\mathcal{S}$ containing $\{u\} \cup L_1$ such that $\widetilde{X_1}$ is a special subgeometry. Since the set of lines of $\mathcal{S}$ through x contained in X_1 is a subspace of $\mathcal{L}_x$ of dimension $n-1$, there exists a line L_2 through x not contained in X_1. Let X_2 denote the unique subspace of $\mathcal{S}$ containing $\{u\} \cup L_2$ such that $\widetilde{X_2}$ is a special subgeometry. Let S_i, $i \in \{1, 2\}$, denote the set of lines through x contained in X_i. Then S_i is an $(n-1)$-dimensional subspace of $\mathcal{L}_x$. If $L \in S_1 \cap S_2$, then $\widetilde{X_1}$ and $\widetilde{X_2}$ are two distinct special subgeometries containing $\{u\} \cup L$, in contradiction with Theorem 4.39(3). Hence, $S_1 \cap S_2 = \emptyset$, implying that the projective space $\mathcal{L}_x$ has dimension at least $2n-1$. $\square$

Lemma 4.41. *Let $\mathcal{G}$ be a special subgeometry, x a point of $\mathcal{G}$ and S_x the $(n-1)$-dimensional subspace of $\mathcal{L}_x$ determined by the lines through x contained in $\mathcal{G}$. If F is a convex subspace of diameter $\delta \in \{0, 1, \dots, n-1\}$ through x such that all lines of F through x belong to S_x, then $F \subseteq \mathcal{G}$.*

Proof. This is obviously the case if F is the singleton $\{x\}$ or a line. So, suppose the diameter δ of F is at least 2. By Lemma 4.15, $\widetilde{F}$ is a dual polar space of rank δ and so the subspace of $\mathcal{L}_x$ induced by F is $(\delta-1)$-dimensional. There exist thus lines $L_1, L_2, \dots, L_\delta$ through x generating the subspace of $\mathcal{L}_x$ induced by F. For every $i \in \{1, 2, \dots, \delta\}$, let F_i be the convex subspace $\langle L_1, L_2, \dots, L_i\rangle$. Then $F_i \subseteq F$ has diameter i. We show by induction on i that $F_i \subseteq \mathcal{G}$. This is clearly true if $i = 1$. Suppose now that $F_i \subseteq \mathcal{G}$ for some $i \in \{1, 2, \dots, \delta-1\}$. Since F_i is a convex subspace of diameter i, there exists by Corollary 4.5 a point $y_i \in F_i$ at distance i from x. If $z_{i+1} \in L_{i+1} \setminus \{x\}$, then $F_{i+1} = \langle L_{i+1}, F_i\rangle = \langle z_{i+1}x, x, y_i\rangle = \langle z_{i+1}, y_i\rangle$, where the last equality follows from the fact that $x = \pi_{F_i}(z_{i+1})$ is on a shortest path between z_{i+1} and $y_i \in F_i$. Since $\mathrm{d}(z_{i+1}, y_i) = \mathrm{d}(z_{i+1}, x) + \mathrm{d}(x, y_i) = i + 1 \le \delta \le n-1$, we know that $F_{i+1} = \langle y_i, z_{i+1}\rangle \subseteq \mathcal{G}$.

In particular, we have $F = F_\delta \subseteq \mathcal{G}$. $\square$

Lemma 4.42. *Let x be a point of $\mathcal{S}$ and S a subspace of dimension $n-1$ from $\mathcal{L}_x$. Then there exists a unique special subgeometry $\mathcal{G}$ through x such that the lines of S are precisely the lines through x contained in $\mathcal{G}$.*

Proof. Let S' be a hyperplane of S in $\mathcal{L}_x$ and $L \in S \setminus S'$. By Lemma 4.17, there exists a unique max F through x such that the lines of S' are the lines of F through x. By Corollary 4.5, there exists a point y in F at maximal distance $n-1$ from x. Any special geometry containing the lines of S must also contain F by Lemma 4.41 and hence also $L \cup \{y\}$. This in combination with Theorem 4.39(3) allows us to conclude that any special subgeometry containing the lines of S must coincide with the unique special subgeometry $\mathcal{G}$ containing $L \cup \{y\}$.

Conversely, the special subgeometry $\mathcal{G}$ contains x and y, thus also $F = \langle x, y\rangle$ and all lines of $S' = S(x, y)$ by Lemma 4.10. The set of lines of $\mathcal{G}$ through x, which is an $(n-1)$-dimensional subspace of $\mathcal{L}_x$, therefore coincides with the subspace S of $\mathcal{L}_x$ generated by S' and L. $\square$

5 Proof of Theorem 1.2

In this section, we suppose that $\mathcal{S}$ is a finite near $2d$-gon, $d \ge 3$, satisfying the following two properties:

(1) Every line of $\mathcal{S}$ has at least three points and every two points at distance 2 have at least three common neighbours (in particular, $\mathcal{S}$ is dense).

(2) If H is a hex, then $\widetilde{H}$ is a dual polar space (of rank 3).

If not all lines of $\mathcal{S}$ are incident with the same number of points, then by [5, Theorem 1], $\mathcal{S}$ is the direct product of two near polygons $\mathcal{S}_1$ and $\mathcal{S}_2$ of diameter at least 1, i.e. the collinearity graph of $\mathcal{S}$ is the cartesian product of the collinearity graphs of $\mathcal{S}_1$ and $\mathcal{S}_2$. One can then find two points x and y in $\mathcal{S}$ such that $\mathrm{d}(x,y)=2$ and $|\Gamma_1(x)\cap\Gamma_1(y)|=2$, namely $x=(x_1,x_2)$ and $y=(y_1,y_2)$ where x_i,y_i with $i\in\{1,2\}$ are two points of $\mathcal{S}_i$ for which $x_i\sim y_i$. This would however violate the claim that every two points at distance 2 have at least three common neighbours. So, there exists an $s\in\mathbb{N}\setminus\{0,1\}$ such that every line of $\mathcal{S}$ is incident with precisely $s+1$ points. By Tits' classification of polar spaces (see Section 2), we have:

Lemma 5.1. *If $\mathcal{S}$ is a dual polar space, then $\mathcal{S}$ is isomorphic to $DW(2d-1,r)$, $DQ(2d,r)$, $DQ^-(2d+1,r)$, $DH(2d-1,r^2)$ or $DH(2d,r^2)$ for some prime power r.*

In the sequel, we suppose that $\mathcal{S}$ is not a dual polar space and derive a contradiction. We take a convex subspace F of smallest possible diameter k such that $\widetilde{F}$ is not a dual polar space. Then $4\le k\le d$ and $\widetilde{F}$ satisfies the properties (P1), (P2), (P3) of Section 4.

Applying Lemma 5.1 to the hexes of $\mathcal{S}$, we find:

Lemma 5.2. *If H is a hex, then one of the following cases occurs:*

(1) *$s=q$ for some prime power and $\widetilde{H}\cong DW(5,q)$ or $\widetilde{H}\cong DQ(6,q)$;*

(2) *$s=q^2$ for some prime power and $\widetilde{H}\cong DQ^-(7,q)$;*

(3) *$s=q^{\frac{1}{2}}$ for some prime power q and $\widetilde{H}\cong DH(5,q)$;*

(4) *$s=q^{\frac{3}{2}}$ for some prime power q and $\widetilde{H}\cong DH(6,q)$.*

Lemma 5.3. *$\widetilde{F}$ is a regular near polygon.*

Proof. For any nonempty convex subspace G of diameter at least 2 of $\widetilde{F}$, $\widetilde{G}$ is a dense near polygon and so has order (s,t_G) for some $t_G\in\mathbb{N}$ by [5, Lemma 19]. In particular, $\widetilde{F}$ has order (s,t_F). By Lemma 4.10, it suffices to prove that $t_{G_1}=t_{G_2}$ for any two convex subspaces G_1 and G_2 of $\widetilde{F}$ of the same diameter $i\in\{2,3,\dots,k-1\}$.

Let Q_j, $j\in\{1,2\}$, be a quad contained in G_j. Then $t_{G_j}+1=\frac{t_{Q_j}^i-1}{t_{Q_j}-1}$ as $\widetilde{G_j}$ is a dual polar space of rank i (for which each local space is a projective space of dimension $i-1$). So, it suffices to prove the equality $t_{Q_1}=t_{Q_2}$. As $\widetilde{F}$ is connected, it suffices to prove the equality $t_{Q_1}=t_{Q_2}$ in case Q_1 and Q_2 have a common point x.

Let Q_3 be a quad through x that intersects Q_1 and Q_2 in lines. As $H_1=\langle Q_1,Q_3\rangle$ and $H_2=\langle Q_2,Q_3\rangle$ are hexes, we indeed know by Lemma 5.2 that $t_{Q_1}=t_{Q_3}=t_{Q_2}$. $\square$

Let s,t,t_i, $i\in\{0,1,\dots,k\}$, be the parameters of $\widetilde{F}$ regarded as a regular near polygon. Combining the above with Lemma 4.40, we find:

Lemma 5.4. *We have*

- $s \in \{q, q^2, q^{\frac{1}{2}}, q^{\frac{3}{2}}\}$ *for some prime power* q*;*
- $t_i + 1 = \frac{q^i - 1}{q-1}$ *for every* $i \in \{0, 1, \ldots, k-1\}$*;*
- $t_k + 1 = \frac{q^e - 1}{q-1}$*, where* $e \geq 2k$ *is some integer.*

We now use the theory of distance-regular graphs and regular near polygons to derive a contradiction. Among other results, we will rely on some inequalities regarding the parameters of regular near polygons obtained by Brouwer & Wilbrink [5], Neumaier [10] and Hiraki & Koolen [9].

Lemma 5.5. *The integer* k *must be odd.*

Proof. If k is even, then the bound

$$t_k + 1 \leq (s^2 + 1)(t_{k-1} + 1)$$

of Brouwer and Wilbrink [5, p. 161] in combination with Lemma 5.4 implies that

$$\frac{q^{2k} - 1}{q - 1} \leq t_k + 1 \leq (s^2 + 1)(t_{k-1} + 1) = (s^2 + 1)\frac{q^{k-1} - 1}{q - 1} < (s^2 + 1)\frac{q^k - 1}{q - 1}.$$

This implies that $q^{\frac{k}{2}} < s$, in contradiction with the fact that $k \geq 4$ and $s \in \{q, q^2, q^{\frac{1}{2}}, q^{\frac{3}{2}}\}$. $\square$

Lemma 5.6. *We have* $s = q^2$ *and* $t_k + 1 = \frac{q^{2k} - 1}{q - 1}$.

Proof. The bound

$$t_k + 1 \leq \frac{s^k + 1}{s^{k-2} + 1}(t_{k-1} + 1 + s^{k-2})$$

of Neumaier [10, Theorem 3.1] in combination with Lemma 5.4 implies that

$$q^{2k-1} + q^{2k-2} < \frac{q^{2k} - 1}{q - 1} \leq t_k + 1 \leq \frac{s^k + 1}{s^{k-2} + 1}(t_{k-1} + 1 + s^{k-2})$$

$$< s^2\left(\frac{q^{k-1} - 1}{q - 1} + s^{k-2}\right) < s^2(q^{k-1} + s^{k-2}).$$

If $s \in \{q, q^{\frac{1}{2}}, q^{\frac{3}{2}}\}$, then this implies that $q^{2k-1} + q^{2k-2} < q^{k+2} + q^{\frac{3k}{2}}$, in contradiction with $k + 2 < 2k - 1$ and $\frac{3k}{2} \leq 2k - 2$. Hence, $s = q^2$.

If $t_k + 1 \neq \frac{q^{2k} - 1}{q - 1}$, then the above-mentioned Neumaier bound in combination with Lemma 5.4 implies that

$$q^{2k} + q^{2k-1} + \cdots + q + 1 \leq t_k + 1 \leq \frac{s^k + 1}{s^{k-2} + 1}(t_{k-1} + 1 + s^{k-2}) < s^2(t_{k-1} + 1 + s^{k-2})$$

$$\leq q^4(1 + q + \cdots + q^{k-2} + q^{2k-4}),$$

clearly a contradiction. Hence, $t_k + 1 = \frac{q^{2k} - 1}{q - 1}$. $\square$

Lemma 5.7. *We have $k \in \{5,7\}$. If $k=7$, then $q \in \{2,3,4\}$.*

Proof. Put $\tau := \frac{2k+1}{2k-2}$. Then the inequality

$$\frac{t_k+1}{t_{k-1}+1} < s(2s^\tau + \frac{1}{2})^2 + 1$$

of Hiraki and Koolen [9, Theorem 1] in combination with Lemmas 5.4 and 5.6 implies that

$$q^{k+1} + 1 < \frac{q^{2k}-1}{q^{k-1}-1} = \frac{t_k+1}{t_{k-1}+1} < s(2s^\tau + \frac{1}{2})^2 + 1 = q^2(2q^{2\tau} + \frac{1}{2})^2 + 1,$$

or $q^{\frac{k-1}{2}} - 2q^{2\tau} - \frac{1}{2} < 0$. If $k \geq 9$, then $q^{\frac{k-1}{2}} - 2q^{2\tau} - \frac{1}{2} \geq q^4 - 2q^{\frac{19}{8}} - \frac{1}{2} = q^{\frac{19}{8}}(q^{\frac{13}{8}} - 2) - \frac{1}{2} > 0$, a contradiction. Hence, $k \in \{5,7\}$ by Lemma 5.5. If $k=7$, then $q^3 - 2q^{\frac{5}{2}} - \frac{1}{2} < 0$ implies that $q \in \{2,3,4\}$. $\square$

By Lemmas 5.4, 5.6 and 5.7, the following possibilities for the parameters remain:

(1) $k=5$, $t_2 = q$, $s = q^2$, $t_3 = q^2+q$, $t_4 = q^3+q^2+q$ and $t_5 = q^9+q^8+q^7+q^6+q^5+q^4+q^3+q^2+q$,

(2) $k=7$, $t_2 = q$, $s = q^2$, $t_3 = q^2+q$, $t_4 = q^3+q^2+q$, $t_5 = q^4+q^3+q^2+q$, $t_6 = q^5+q^4+q^3+q^2+q$ and $t_7 = q^{13}+q^{12}+q^{11}+q^{10}+q^9+q^8+q^7+q^6+q^5+q^4+q^3+q^2+q$, where $q \in \{2,3,4\}$.

The collinearity graph Γ of $\widetilde{F}$ is a distance regular graph of diameter k, whose parameters are $a_i = (s-1)(t_i+1)$, $b_i = s(t_k - t_i)$, $c_i = t_i + 1$ for $i \in \{0,1,\dots,k\}$. Now, define polynomials $u_i(x) \in \mathbb{Q}(x)$ in the following recursive way:

$$u_0(x) := 1,\ u_1(x) := \frac{x}{b_0},\ u_{i+1}(x) := \frac{1}{b_i}\Big(-c_i \cdot u_{i-1}(x) + (x-a_i)\cdot u_i(x)\Big),\ i \in \{1,2,\dots,k-1\}.$$

If θ is an eigenvalue of Γ, then $(u_0(\theta), u_1(\theta), \dots, u_k(\theta))$ is called the *standard sequence of Γ corresponding to θ* ([4, Section 4.1]).

Now, the smallest eigenvalue of Γ is equal to $-(t_k+1)$, see Brouwer and Wilbrink [5, p. 160–161]. One easily verifies that the standard sequence of this eigenvalue is equal to $(1, -\frac{1}{s}, \frac{1}{s^2}, \dots, (-\frac{1}{s})^k)$. If n_i, $i \in \{0,1,\dots,k\}$, denotes the constant number of points of F at distance i from a given point of F, then

$$n_i = \frac{s^i \cdot \prod_{j=0}^{i-1}(t-t_j)}{\prod_{j=1}^{i}(t_j+1)}$$

by standard counting (see [5, 7]). By Biggs [1, p. 19], [2, p. 95] (see also [4, Theorem 4.1.4]), the multiplicity of an eigenvalue θ of Γ is equal to

$$\frac{|F|}{\sum_{i=0}^k n_i \cdot (u_i(\theta))^2} = \frac{\sum_{i=0}^k n_i}{\sum_{i=0}^k n_i \cdot (u_i(\theta))^2}.$$

In particular, the multiplicity of the eigenvalue $-(t_k+1)$ is equal to

$$m := \Big(\sum_{i=0}^{k} n_i\Big)\Big(\sum_{i=0}^{k} n_i s^{-2i}\Big)^{-1}.$$

(1) Now, consider the case where $k=5$, $t_2=q$, $s=q^2$, $t_3=q^2+q$, $t_4=q^3+q^2+q$ and $t_5=q^9+q^8+q^7+q^6+q^5+q^4+q^3+q^2+q$. Then one calculates that

$$m = f_1(q) + \frac{f_2(q)}{f_3(q)},$$

where

$$\begin{aligned}
f_1(q) &= q^{18}-q^{14}+3q^{10}-q^9-q^8-4q^6+3q^5+4q^4-2q^3+3q^2-6q-7,\\
f_2(q) &= 9q^{22}+10q^{21}+29q^{20}+44q^{19}+53q^{18}+73q^{17}+105q^{16}+127q^{15}\\
&\quad +161q^{14}+178q^{13}+178q^{12}+179q^{11}+188q^{10}+183q^9+178q^8\\
&\quad +164q^7+120q^6+86q^5+80q^4+59q^3+46q^2+40q+14,\\
f_3(q) &= q^{23}+q^{22}+2q^{21}+3q^{20}+6q^{19}+7q^{18}+10q^{17}+12q^{16}+15q^{15}\\
&\quad +17q^{14}+20q^{13}+20q^{12}+21q^{11}+20q^{10}+19q^9+18q^8+17q^7\\
&\quad +13q^6+10q^5+8q^4+6q^3+4q^2+4q+2.
\end{aligned}$$

Notice that all the nonzero coefficients of the polynomials $f_2(q)$ and $f_3(q)$ are positive. If C_i, $i\in\{0,1,\ldots,22\}$, denotes the coefficient of q^i in the polynomial $f_2(q)$ and D_i, $i\in\{0,1,\ldots,23\}$, denotes the coefficient of q^i in the polynomial $f_3(q)$, then we have that $C_i \le \frac{44}{3}D_{i+1}$ for every $i\in\{0,1,\ldots,22\}$. This implies that $0<f_2(q)<f_3(q)$ for every $q\ge 15$. So, m cannot be integral if $q\ge 15$. If $q<15$, i.e. $q\in\{2,3,4,5,7,8,9,11,13\}$, then one can also easily verify that $\frac{f_2(q)}{f_3(q)}$ and hence also m is not integral. So, the case $k=5$ cannot occur.

(2) Consider now the case $k=7$, $t_2=q$, $s=q^2$, $t_3=q^2+q$, $t_4=q^3+q^2+q$, $t_5=q^4+q^3+q^2+q$, $t_6=q^5+q^4+q^3+q^2+q$, $t_7=q^{13}+q^{12}+q^{11}+q^{10}+q^9+q^8+q^7+q^6+q^5+q^4+q^3+q^2+q$, where $q\in\{2,3,4\}$. Putting $m_q=m$, we find

$$\begin{aligned}
m_2 &= 66144536+\frac{105089758162492}{145959280562129}\notin\mathbb{N},\\
m_3 &= 2538429414694+\frac{2340656569502919078170245}{2885075998038235811542463}\notin\mathbb{N},\\
m_4 &= 4502504861592725+\frac{2932739491798122360242337990887}{2946142492535696835096707829397}\notin\mathbb{N}.
\end{aligned}$$

None of these values is thus integral, and so this case cannot occur. We thus conclude that $\mathcal{S}$ must be a dual polar space.

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References

[1] N. Biggs. Intersection matrices for linear graphs. pp. 15–23 in *Combinatorial Mathematics and its Applications* (Proc. Conf., Oxford, 1969). Academic Press, 1971.

[2] N. Biggs. *Finite groups of automorphisms.* London Mathematical Society Lecture Note Series 6. Cambridge University Press, 1971.

[3] A. E. Brouwer and A. M. Cohen. Local recognition of Tits geometries of classical type. *Geom. Dedicata* 20 (1986), 181–199.

[4] A. E. Brouwer, A. M. Cohen and A. Neumaier. *Distance-regular graphs.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 18. Springer-Verlag, 1989.

[5] A. E. Brouwer and H. A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata* 14 (1983), 145–176.

[6] P. J. Cameron. Dual polar spaces. *Geom. Dedicata* 12 (1982), 75–85.

[7] B. De Bruyn. *Near polygons.* Birkhäuser, 2006.

[8] B. De Bruyn. *An introduction to incidence geometry.* Birkhäuser, 2016.

[9] A. Hiraki and J. Koolen. A generalization of an inequality of Brouwer-Wilbrink. *J. Combin. Theory Ser. A* 109 (2005), 181–188.

[10] A. Neumaier. Krein conditions and near polygons. *J. Combin. Theory Ser. A* 54 (1990), 201–209.

[11] S. E. Payne and J. A. Thas. *Finite generalized quadrangles.* Second edition. EMS Series of Lectures in Mathematics. European Mathematical Society, 2009.

[12] S. Shad and E. Shult. The near n-gon geometries. Unpublished manuscript.

[13] E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata* 9 (1980), 1–72.

[14] J. Tits. *Buildings of spherical type and finite BN-pairs.* Lecture Notes in Mathematics 386. Springer-Verlag, 1974.

[15] J. Tits. A local approach to buildings. pp. 519–547 in *The geometric vein.* Springer, 1981.